Exact distributions of the number of distinct and common sites visited by N independent random walkers

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We study the number of distinct sites $S_N(t)$ and common sites $W_N(t)$ visited by N independent one dimensional random walkers, all starting at the origin, after t time steps. We show that these two random variables can be mapped onto extreme value quantities associated to N independent random walkers. Using this mapping, we compute exactly their probability distributions $P_N^d(S,t)$ and $P_N^d(W,t)$ for any value of N in the limit of large time t, where the random walkers can be described by Brownian motions. In the large N limit one finds that $S_N(t)/\sqrt{t} \propto 2\sqrt{\log N} + \widetilde{s}/(2\sqrt{\log N})$ and $W_N(t)/\sqrt{t} \propto \widetilde{w}/N$ where $\widetilde{s}$ and $\widetilde{w}$ are random variables whose probability density functions (pdfs) are computed exactly and are found to be non trivial. We verify our results through direct numerical simulations.

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In elementary set theory, two fundamental concepts are the *union* and the *intersection* of a number of N sets. While the union consists of all *distinct* elements of the collection of sets, the intersection consists of *common* elements of all the sets. These two notions appear naturally in everyday life: for example the area of common knowledge or the whole range of different interests amongst the members of a society would define respectively its stability and activity. In an habitat of N animals, the union of the territories covered by different animals sets the geographical range of the habitat, while the intersection refers to the common area (e. g. a water body) frequented by all animals.

In statistical physics, these two objects are modeled respectively by the number of distinct and common sites visited by N random walkers (RWs). The knowledge about the number of distinct sites has applications ranging from the annealing of defects in crystals [1, 2] and relaxation processes [3–6] to the spread of populations in ecology [7, 8] or to the dynamics of web annotation systems [9]. Similarly the knowledge about the common area frequented by endangered animals is very useful for their daily health caring. Likewise, in the energy transport through a series of independent disordered samples, the energy output will depend on the number of energy levels common to all these materials.

Dvoretzky and Erdös [10] first studied the average number of distinct sites $\langle S_1(t)\rangle$ visited by a single t-step RW in d-dimensions, subsequently studied in [11–13]. Larralde et al. generalized this to N independent, t-step walkers moving on a d-dimensional lattice [14]. They found three regimes of growth (early, intermediate and late) for the average number of distinct sites $\langle S_N(t)\rangle$ as a function of time. These three regimes are separated by two N-dependent times scales [14]. In particular they showed that in $d = 1$ and $t \gg \sqrt{\log N}$, $\langle S_N(t)\rangle \propto \sqrt{4D\ t\ \log N}$ where D is the diffusion constant of a single walker. Recently Majumdar and Tamm [15] studied the complementary quantity, namely the number of common sites $W_N(t)$ visited by N walkers, each of t steps, and found analytically a rich asymptotic late time growth of the average $\langle W_N(t)\rangle$. They showed that in the $(N-d)$ plane there are three distinct phases separated by two critical lines $d = 2$ and $d_c(N) = 2N/(N-1)$, with $\langle W_N(t)\rangle \sim t^\nu$ at late times where the growth exponent $\nu = d/2$ (for $d < 2$), $\nu = N - d(N-1)/2$ [for $2 < d < d_c(N)$] and $\nu = 0$ [for $d > d_c(N)$] (see also [16]). In particular, in $d = 1$, $\langle W_N(t)\rangle \sim \sqrt{4Dt}$ where the prefactor depends on N. However, most of these studies were limited to the *average* number of distinct or common sites, and there exists virtually no information about their full probability distributions, e.g. the probabilities $P_N^d(S,t)$ that $S_N(t) = S$ and $P_N^c(W,t)$ that $W_N(t) = W$.

Computing these distributions for general d-dimensional space is highly non trivial. Indeed, although the N walkers are independent, conditioning their trajectories to a given number of distinct (or common) visited sites introduces strong effective correlations between them. In $d = 1$, we show here that these random variables $S_N(t)$ and $W_N(t)$ can be mapped onto extreme values (nearest and furthest displacements) associated to N independent walkers. This connection to extreme value statistics (EVS) allows us to compute $P_N^d(S,t)$ and $P_N^c(W,t)$ exactly for t large and arbitrary N. We show that the induced correlations between the walkers persist even for $N \to \infty$ where the limiting distributions are not given by EVS of independent random variables, as erroneously argued in the previous study of $S_N(t)$ [14].

We consider N independent and identical t-step RWs $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ on a 1-d lattice, all starting at the origin. For convenience, we set the diffusion constant of the walkers $D = \frac{1}{2}$. Distinct sites are those that are visited at least once by at least one of the N walkers [14], while common sites correspond to sites visited individually at least once by all the N walkers [15]. We denote by M_i and m_i respectively the maximum and the min-

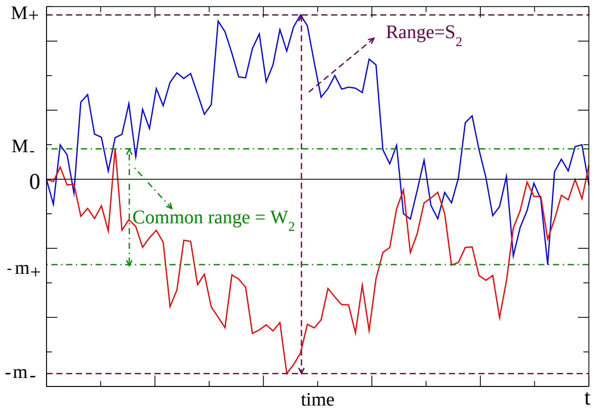

FIG. 1. (Color Online) Schematic diagram of 2 independent RWs, where M_+, M_-, m_+, m_- and S_2, W_2 are shown (1, 2).

imum displacements of the $i^{\rm th}$ walker x_i up to time t. The number of distinct sites visited, S_N [17], is then the sum of the range on the positive (+ve) side, M_+, and the range on the negative (-ve) side m_- (see Fig. 1):

$$S_N = M_+ + m_- \; , \; M_+ = \max_{1\le i\le N} M_i \; , \; m_- = -\min_{1\le i\le N} m_i \; . \tag{1}$$

Similarly, the number of common sites visited, W_N, is the common span on the +ve axis plus the common span m_+ on the -ve axis:

$$W_N = M_- + m_+ \; , M_- = \min_{1\le i\le N} M_i \; , \; m_+ = -\max_{1\le i\le N} m_i \; . \tag{2}$$

Eqs. (1) and (2) establish a precise connection between S_N and W_N and the EVS of N independent RW's.

In the limit of large t, the lattice RWs converge to Brownian motions (BMs). Hence for large t, the probability distributions $P_N^d(S,t)$ and $P_N^c(W,t)$ take the scaling form

$$P_N^d(S,t) = \frac{1}{\sqrt{2t}} p_N^d\left(\frac{S}{\sqrt{2t}}\right) \; , \; P_N^d(W,t) = \frac{1}{\sqrt{2t}} p_N^d\left(\frac{W}{\sqrt{2t}}\right) \tag{3}$$

where $p_N^d(s)$ is the probability density function (pdf) of the span or range, $s = S/\sqrt{2t}$, and $p_N^c(w)$ is the pdf of the common span or common range, $w = W/\sqrt{2t}$, for N independent BMs (see Fig. 1) on the unit time interval [18]. The rescaled quantities $S_N/\sqrt{2t}$ and $W_N/\sqrt{2t}$ in (3) are given by (1) and (2) where $M_\pm, m_\pm$ are replaced by their counterparts $\widetilde{M}_\pm = M_\pm/\sqrt{2t}$ and $\widetilde{m}_\pm = m_\pm/\sqrt{2t}$ corresponding to N independent BMs on the unit time interval.

It is useful to summarize our main results. We obtain exactly, for any N, the pdfs $p_N^d(s)$ and $p_N^c(w)$ as presented in (12) and (15) along with (8) and (9). The moments can also be computed explicitly [19]. The tails of the pdfs can be derived explicitly:

$$p_N^d(s) \sim \begin{cases} a_N s^{-5} \exp\left[-N\pi^2/(4s^2)\right] \; , \; s \to 0 \; , \\ b_N \exp\left(-s^2/2\right) \; , \; s \to \infty \; , \end{cases} \tag{4}$$

and

$$p_N^c(w) \sim \begin{cases} c_N \, w \; , \; w \to 0 \\ d_N w^{1-N} \exp\left(-N\, w^2\right) \; , \; w \to \infty \; , \end{cases} \tag{5}$$

where a_N, b_N, c_N and d_N are computable constants (see below). For $N \to \infty$, one finds that both pdfs approach a non trivial limiting form

$$p_N^d(s) \sim 2\sqrt{\log N}\, \mathcal{D}\left(2\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)\right) \; ,$$
$$\mathcal{D}(\widetilde{s}) = 2\, e^{-\widetilde{s}} K_0(2\, e^{-\widetilde{s}/2}) \; , \tag{6}$$

where $K_n(x)$ denote the modified Bessel functions, and

$$p_N^c(w) = N\, \mathcal{C}\left(Nw\right) , \; \mathcal{C}(\widetilde{w}) = \frac{4}{\pi}\, \widetilde{w}\, e^{-\frac{2}{\sqrt{\pi}}\widetilde{w}} \; , \; \widetilde{w} > 0 \, . \tag{7}$$

Note that $\mathcal{D}(\tilde{s})$ (6) is not the Gumbel distribution, as it was initially argued in [14]. Remarkably the same distribution $\mathcal{D}(\tilde{s})$ also appears as the limiting distribution of the maximum of a large collection of logarithmically correlated random variables on a circle [20]. We check indeed $\int_{-\infty}^{\tilde{s}} \mathcal{D}(\tilde{s}')d\tilde{s}' = 2e^{-\tilde{s}/2}K_1(2e^{-\tilde{s}/2})$, as obtained in [20]. Incidentally, logarithmically correlated random variables have been the subject of several recent studies [20–22] because they exhibit freezing phenomena, akin to the replica symmetry breaking scenario found in mean field spin glass models [23]. As a byproduct of our computation, we show that $\mathcal{D}(\tilde{s})$ is the convolution of two independent Gumbel distributions.

We start by computing the joint cumulative distribution functions (jcdf) $\mathbf{P}_d\left(l_1, l_2\right) = \Pr.\left(\widetilde{M}_+ \le l_1, \widetilde{m}_- \le l_2\right)$, relevant for $p_N^d(s)$ and the jcdf $\mathbf{P}_c\left(j_1, j_2\right) = \Pr.\left(\widetilde{M}_- \ge j_1, \widetilde{m}_+ \ge j_2\right)$ relevant for $p_N^c(w)$. Since all the N BMs are identical and independent, $\mathbf{P}_d\left(l_1, l_2\right) = g^N(l_1, l_2)$, where $g(l_1, l_2) = \Pr.(\widetilde{M} \le l_1, \widetilde{m} \ge -l_2)$ is the jcdf of the maximum $\widetilde{M}$ and the minimum $\widetilde{m}$ for a *single* BM on the unit time interval. It can be computed by the standard method of images [24]:

$$g(l_1, l_2) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{(2\,n+1)\pi l_2}{l_1+l_2}\right) e^{-\left(\frac{(n+\frac{1}{2})\pi}{l_1+l_2}\right)^2} \, . \tag{8}$$

Similarly, $\mathbf{P}_c\left(j_1, j_2\right) = h^N(j_1, j_2)$ where $h(j_1, j_2) = \Pr.\left(\widetilde{M} \ge j_1, \widetilde{m} \le -j_2\right)$ reads:

$$h(j_1, j_2) = 1 - \mathrm{erf}\left(j_1\right) - \mathrm{erf}\left(j_2\right) + g(j_1, j_2), \tag{9}$$

where $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-y^2}dy$, $\mathrm{erf}\left(j_1\right) = \Pr.(\widetilde{M} \le j_1)$ and $\mathrm{erf}\left(j_2\right) = \mathrm{Prob}(\widetilde{m} \ge -j_2)$. From the joint pdf

$\frac{\partial^2 \mathbf{P}_d(l_1,l_2)}{\partial l_1 \partial l_2}$ and using (1), we obtain

$$p_N^d(s) = \int_0^\infty dl_1 \int_0^\infty dl_2 \ \delta(s - l_1 - l_2) \frac{\partial^2 g^N}{\partial l_1 \partial l_2} \ , \qquad (10)$$

with $g \equiv g(l_1, l_2)$. Similarly, from the joint pdf $\frac{\partial^2 \mathbf{P}_c(j_1,j_2)}{\partial j_1 \partial j_2}$ and using (2) we obtain,

$$p_N^c(w) = \int_0^\infty dj_1 \int_0^\infty dj_2 \ \delta(w - j_1 - j_2) \frac{\partial^2 h^N}{\partial j_1 \partial j_2} \ , \qquad (11)$$

with $h \equiv h(j_1, j_2)$. For small values of N, the double integrals in (10) and (11) can be performed explicitly and numerical simulations confirm these exact results [19] . Below we provide a physical interpretation of these formulas (10, 11) and perform, separately, their asymptotic analysis both for small and large arguments. We also analyze their limiting form for $N \to \infty$.

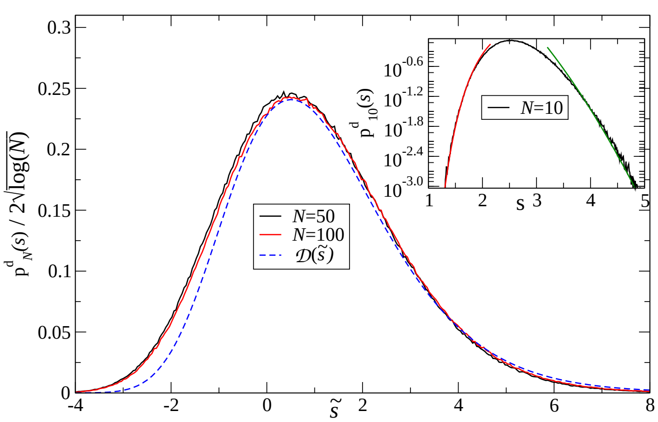

FIG. 2. (Color online) Plot of $p_N^d(s)/(2\sqrt{\log N})$ as a function of $\widetilde{s} = 2\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)$. The dotted line indicates the exact asymptotic results for $N \to \infty$, $\mathcal{D}(\widetilde{s})$ in (6). **Inset:** Plot of $p_{10}^d(s)$, obtained from simulation, compared with its asymptotic behavior (4).

Distinct sites : To find the tails of $p_N^d(s)$ at small and large s for finite N, we rewrite (10) as

$$p_N^d(s) = \int_0^s dl_2 \Psi_d(s - l_2, l_2) \quad \text{where} \qquad (12)$$

$$\Psi_d(l_1, l_2) = N \ g^{N-1} \frac{\partial^2 g}{\partial l_1 \partial l_2} + N(N-1) g^{N-2} \frac{\partial g}{\partial l_1} \frac{\partial g}{\partial l_2} \ .$$

We interpret the two contributions in $\Psi_d(l_1, l_2)$ as follows [19]: the first term corresponds to a configuration where one particle explores a region $[-l_2, s - l_2]$ (we call it a box) of size s in unit time interval, such that its maximum is at $s - l_2$ and minimum is at $-l_2$, while all the other $(N-1)$ particles stay inside this box. On the other hand, the second term corresponds to a configuration where two particles create, in a different way, the same box $[-l_2, s - l_2]$ of size s: one of the two particles has its maximum at $s - l_2$ and minimum larger than $-l_2$ while the second particle has its minimum at $-l_2$ and maximum below $s - l_2$ and all other $(N-2)$ particles stay strictly inside this box.

When $s \to 0$ in (12), one can replace $g(l_1, l_2)$ (8) by its asymptotic behavior when $l_1, l_2 \to 0$ where $g(l_1, l_2) \sim \frac{4}{\pi} \sin\left(\frac{\pi \ l_2}{l_1 + l_2}\right) e^{-\frac{\pi^2}{4(l_1+l_2)^2}}$. Inserting it in (12), we see that both terms in (12) contribute equally. After integration over l_2, one then obtains the result announced in (4) for $s \to 0$ with $a_N = 4\pi^{3/2} N(N-1) \left(\frac{4}{\pi}\right)^{N-2} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})}$, where $\Gamma(x)$ is the Gamma function. To perform the large s asymptotic of $p_N^d(s)$ we use the Poisson summation formula: $g(l_1, l_2) = \sum_{m=0}^{\infty} (-1)^m \left[\mathrm{erf}\left[m(l_1 + l_2) + l_1\right] + \mathrm{erf}\left[m(l_1 + l_2) + l_2\right]\right]$. We use this form to evaluate the integrand in (12) in the limit $s \to \infty$. We see that the first term in (12), which corresponds to create a box $[-l_2, s - l_2]$ with one particle, decreases as $e^{-(s+l_2)^2} e^{-l_2^2}$ whereas the second term where the same box is created by two particles decreases as $e^{-(s-l_2)^2} e^{-l_2^2}$. Since l_2 is always +ve, the two particles term wins over the one particle term when $s \to \infty$: this is physically understandable because creating a very large span with two particles is more likely than creating the same one with a single particle. It also follows from this analysis that the integral over l_2 in (12) is dominated by $l_2 \sim \mathcal{O}(s)$, which yields finally the large s behavior announced in (4) with $b_N = 2N(N-1)/\sqrt{\pi}$. In Fig. 2 we verify that the small and large s asymptotics of $p_N^d(s)$ given in (4), for $N = 10$, describe very well, without any fitting parameter, the distribution obtained from direct simulation, without any fitting parameter.

What happens for large N ? The typical scale of the fluctuations of $S_N/\sqrt{2t}$ can be estimated from the relations with EVS (1). The variables $\widetilde{M}_i$'s, with $i = 1, \cdots, N$, which are the maxima of the i^{th} BM on the unit interval, are i.i.d. variables. Their common pdf is known to be a half-Gaussian, $p(M) = (2/\sqrt{\pi}) e^{-M^2}$, $M > 0$. The same holds for the variables $-\widetilde{m}_i$'s. Hence, for large N, standard results of EVS [25] state that the typical value of $\widetilde{M}_+ = \max_{1 \leq i \leq N} \widetilde{M}_i$ is $\mathcal{O}(\sqrt{\log N})$ while its fluctuations are of order $1/\sqrt{\log N}$ and governed by a Gumbel distribution. The same also holds for $\widetilde{m}_- = -\min_{1 \leq i \leq N} \widetilde{m}_i$. For large N, these two extremes become uncorrelated as the global maximum and global minimum are most likely reached by two independent walkers. Hence one gets

$$g^N \left[\mu_N + \tfrac{\tilde{l}_1}{2\mu_N}, \mu_N + \tfrac{\tilde{l}_2}{2\mu_N}\right] \underset{N \to +\infty}{\longrightarrow} e^{-e^{-\tilde{l}_1}} e^{-e^{-\tilde{l}_2}} \qquad (13)$$

with $\mu_N = \sqrt{\log N}$. Inserting (13) in (10) with $\widetilde{s} = 2\mu_N(s - 2\mu_N)$ one finds

$$p_N^d(s) \sim 2\sqrt{\log N} \int_{-\infty}^{\infty} d\widetilde{l}_2 \ e^{-\widetilde{s}} e^{-e^{-\tilde{l}_2}} e^{-e^{-(\widetilde{s} - \tilde{l}_2)}} \ , \qquad (14)$$

which can be evaluated explicitly to give (6). In Fig. 2 we plot $p_N^d(s)/2\sqrt{\log N}$ against $\widetilde{s}$ for $N = 50$ and 100. They

show a relatively good agreement with the exact result $\mathcal{D}(\tilde{s})$ after an overall shift of order $\mathcal{O}(1/\log N)$ along the x-axis, thus revealing, as expected, a slow convergence towards the asymptotic result. In [14] the authors argued that the limiting distribution should be a Gumbel distribution, overlooking the fact that it is actually the *convolution* of two Gumbel distributions, as in (14). In particular, for large $\tilde{s}$, $\mathcal{D}(\tilde{s}) \sim \tilde{s}e^{-\tilde{s}}$, while the Gumbel distribution decays as a pure exponential.

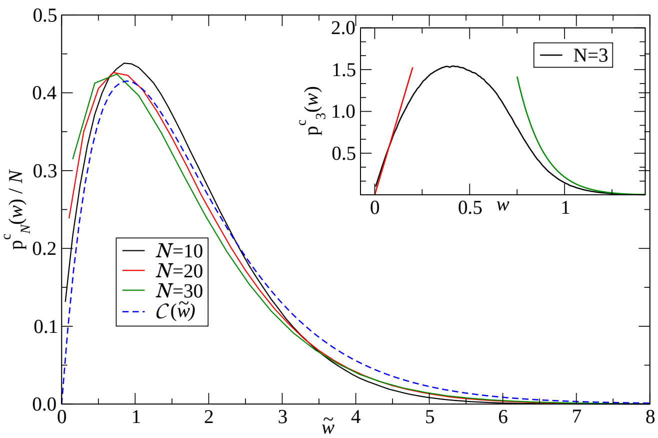

FIG. 3. (Color online) Plot of $p^c_N(w)/N$ as a function of $\widetilde{w} = Nw$. The dotted line indicates the exact asymptotic results for $N \to \infty$, $\mathcal{C}(\widetilde{w})$ in (7). **Inset:** Plot of $p^d_3(w)$, obtained from simulation, compared with its asymptotic behavior (5).

Common sites : To find the small and large w asymptotics of $P^c_N(w)$ we write (11) as

$$p^c_N(w) = \int_0^w dj_2 \Psi_c(w - j_2, j_2) \quad \text{where} \qquad (15)$$

$$\Psi_c(j_1, j_2) = N\ h^{N-1} \frac{\partial^2 h}{\partial j_1 \partial j_2} + N(N-1)h^{N-2} \frac{\partial h}{\partial j_1} \frac{\partial h}{\partial j_2}\ .$$

In (15), one interprets the first term as one single particle creating a common span $[-j_2, w - j_2]$ of size w and the second term as two particles collaboratively creating the same common span (in a unit time interval) [19]. In both cases, the remaining particles are such that their maxima are above $w - j_2$ and their minima are below $-j_2$. When $w \to 0$ in (15), $h(j_1, j_2)$ can be replaced by its asymptotic behavior for small j_1, j_2: $h(j_1, j_2) \sim \left(1 - \frac{2}{\sqrt{\pi}}(j_1 + j_2)\right)$. Integrating then over j_2 in (15) yields the small w behavior in (5) with $c_N = 4N(N-1)/\pi$. Note that for very small w, it is much more likely to create a box of size smaller than w with *two* particles (which occurs with a probability $\propto w^2$) than with a single one [which occurs with probability $\propto \exp\left(-\pi^2/4w^2\right)$]. The former configurations thus dominate for small w.

To get the large w behavior of $p^c_N(w)$, we estimate $h(j_1, j_2)$ for large j_1 (15). This is conveniently done by using the Poisson formula, which yields $h(j_1, j_2) \sim \mathrm{erfc}\,(2j_1 + j_2) + \mathrm{erfc}\,(j_1 + 2j_2)$. This estimate shows that for $w \gg \sqrt{\log N}$, the second term in (15) becomes subdominant compared to the first one. Hence for very large w the leading contribution comes from the first term where we replace $h^{(N-1)}(w - j_2, j_2) \sim [\mathrm{erfc}(w + j_2) + \mathrm{erfc}(2w - j_2)]^{N-1}$ by $\mathrm{erfc}^{(N-1)}(w)$ as one can show that the integral over j_2 in (15) is dominated by the vicinity of $j_2 = 0$ [19]. This leads to the large w behavior in (5) with $d_N = 8N/\pi^{N/2}$. The asymptotic behaviors of $p^c_N(w)$ (5) have been verified numerically for $N = 3$ in Fig. 3.

To obtain the typical scale of $W_N/\sqrt{2t}$ for large N, we use its relation to EVS (2). From standard EVS for i.i.d. random variables [25], we know that $\widetilde{M}_- = \min_{1 \le i \le N} M_i$, where $M_i \ge 0$ and distributed according to a half-Gaussian, is of order $\mathcal{O}(N^{-1})$. Its pdf is given by a Weibull law, which is here an exponential distribution [25]. Indeed one has here $\Pr.(N\widetilde{M}_- \ge x) = e^{-\frac{2}{\sqrt{\pi}}x}$, $x > 0$, as $N \to \infty$. The same holds for $\widetilde{m}_+$, which for large N becomes independent of $\widetilde{M}_-$ as both of them are reached by two independent walkers. Hence, from (2), $NW_N/\sqrt{2t}$ is given by the convolution of two exponential laws:

$$p^c_N(w) \sim N^2(4/\pi)e^{-\frac{2}{\sqrt{\pi}}Nw} \int_0^w dk \sim N\ \mathcal{C}(Nw)\ , \quad (16)$$

with $\mathcal{C}(\widetilde{w})$ as announced in (7). We have also obtained this result [19] by a direct large N expansion of (15). In Fig. 3 we plot $p^c_N(w)/N$ against $\widetilde{w}$ for $N = 10,\ 20$ and 30 and see that they both coincide with the function $\mathcal{C}(\tilde{w})$, although the convergence is rather slow.

Conclusion : We have achieved a complete analytic description of the pdfs of the number of distinct and common sites visited by N independent RWs after t time steps, for large t. We have also obtained interesting limiting distributions (6, 7) in the limit when $N \to \infty$. For distinct sites, we found an intriguing connection with the maximum of logarithmically correlated random variables on a circle [20].

One may wonder about the effects of interactions between the walkers. For instance, one can study non-intersecting (vicious) RWs [27]. An interesting situation is the case where all N walkers start and end at the same point, while staying positive in the time interval $[0, t]$ (watermelons with a wall). In this case, the number of distinct sites $S_N/\sqrt{2t}$ corresponds to the maximal height of these watermelons [28]. For large N, the pdf of $S_N/\sqrt{2t} \propto \sqrt{N}$ properly shifted and scaled, converges to the Tracy-Widom distribution $\mathcal{F}_1$ [29], which describes the fluctuations of the largest eigenvalue of Gaussian orthogonal random matrices. On the other hand, the number of common sites $W_N/\sqrt{2t}$ is related to the maximum of the lower path, the distribution of which is a very interesting open problem [31].

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